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Superrigid Subgroups and Syndetic Hulls in Solvable Lie Groups

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Abstract It is not difficult to see that every group homomorphism from $\mathbb{Z}^k$ to $\mathbb{R}^n$ extends to a homomorphism from $\mathbb{R}^k$ to $\mathbb{R}^n$. We discuss other examples of discrete subgroups Γ of connected Lie groups G, such that the homomorphisms defined on Γ can ("virtually") be extended to homomorphisms defined on all of G. For the case where G is solvable, we give a simple proof that Γ has this property if it is Zariski dense. The key ingredient is a result on the existence of syndetic hulls.

1 What Is a Superrigid Subgroup?

Let us begin with a trivial example of the type of theorem that we will discuss. It follows easily from the fact that a linear transformation can be defined to have any desired action on a basis. (See Sect. 3 for a more complicated proof.)

Proposition 1.1. *Any group homomorphism $\varphi\colon \mathbb{Z}^k \to \mathbb{R}^d$ extends to a continuous homomorphism $\hat{\varphi}\colon \mathbb{R}^k \to \mathbb{R}^d$.*

A superrigidity theorem is a version of this simple proposition in the situation where $\mathbb{Z}^k$, $\mathbb{R}^k$, and $\mathbb{R}^d$ are replaced by more interesting groups. Suppose Γ is a discrete subgroup of a connected Lie group G, and H is some other Lie group. Does every homomorphism $\varphi\colon \Gamma \to H$ extend to a continuous homomorphism $\hat{\varphi}$ defined on all of G?

All of the Lie groups we consider are assumed to be *linear groups*; that is, they are subgroups of $\mathrm{GL}(\ell, \mathbb{C})$, for some ℓ. For example, $\mathbb{R}^d$ can be thought of as a linear group; in particular:

$$\mathbb{R}^3 \cong \begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1}$$

Thus, any homomorphism into $\mathbb{R}^d$ can be thought of as a homomorphism into $\mathrm{GL}(d+1, \mathbb{R})$. The study of homomorphisms into $\mathrm{GL}(d, \mathbb{R})$ or $\mathrm{GL}(d, \mathbb{C})$ is known as *Representation Theory.* Unfortunately, in this much more interesting setting, not all homomorphisms extend.

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Proposition 1.2. *There is a group homomorphism* $\varphi\colon \mathbb{Z} \to \mathrm{GL}(d,\mathbb{R})$ *that does not have a continuous extension to a homomorphism* $\hat{\varphi}\colon \mathbb{R} \to \mathrm{GL}(d,\mathbb{R})$.

Proof. Fix a matrix $A \in \mathrm{GL}(d,\mathbb{R})$, such that $\det A < 0$, and define $\varphi(n) = A^n$, for $n \in \mathbb{Z}$. Then $\varphi(m+n) = \varphi(m)\cdot\varphi(n)$, so φ is a group homomorphism. Since $\det X \neq 0$, for all $X \in \mathrm{GL}(d,\mathbb{R})$, there is no continuous function $\hat{\varphi}\colon \mathbb{R} \to \mathrm{GL}(d,\mathbb{R})$, such that $\hat{\varphi}(0) = \mathrm{Id}$ and $\hat{\varphi}(1) = A$. Hence, φ does not have a continuous extension to $\mathbb{R}$. $\square$

The example shows that we cannot expect $\hat{\varphi}(n)$ to equal $\varphi(n)$ for *all* $n \in \mathbb{Z}$, so we relax the restriction to require equality only when n belongs to some finite-index subgroup of $\mathbb{Z}$. In group theory, it is standard practice to say that a group *virtually* has a property if some finite-index subgroup has the property. In that spirit, we make the following definition.

Definition 1.3. Suppose Γ is a subgroup of G. We say that a homomorphism $\varphi\colon \Gamma \to H$ *virtually extends* to a homomorphism $\hat{\varphi}\colon G \to H$ if there is a finite-index subgroup Γ' of Γ, such that $\varphi(\gamma) = \hat{\varphi}(\gamma)$ for all $\gamma \in \Gamma'$.

The following result is not as trivial as Proposition 1.1.

Proposition 1.4. *Any group homomorphism* $\varphi\colon \mathbb{Z}^k \to \mathrm{GL}(d,\mathbb{R})$ *virtually extends to a continuous homomorphism* $\hat{\varphi}\colon \mathbb{R}^k \to \mathrm{GL}(d,\mathbb{R})$*. Similarly for homomorphisms into* $\mathrm{GL}(d,\mathbb{C})$.

Proposition 1.4 has the serious weakness that it gives no information at all about the image of $\hat{\varphi}$. A superrigidity theorem should state not only that a virtual extension exists, but also that if the image of the original homomorphism φ is well behaved, then the image of the extension $\hat{\varphi}$ is similarly well behaved. For example, we want:

- If $\hat{\varphi}(\Gamma) \subset \mathbb{R}^d$, then $\hat{\varphi}(G) \subset \mathbb{R}^d$ (as embedded in (1)).
- If all the matrices in $\varphi(\Gamma)$ commute with each other, then all the matrices in $\hat{\varphi}(G)$ commute with each other.
- If all of the matrices in $\varphi(\Gamma)$ are upper triangular, then all of the matrices in $\hat{\varphi}(G)$ are upper triangular.

All of these properties, and many more, are obtained by requiring that $\hat{\varphi}(G)$ be contained in the "Zariski closure" of $\varphi(\Gamma)$.

The Zariski closure will be formally defined in Sect. 4. For now, it suffices to have an intuitive understanding:

> The Zariski closure $\overline{\Gamma}$ of a subgroup Γ of $\mathrm{GL}(\ell,\mathbb{C})$ is the "natural" virtually connected subgroup of $\mathrm{GL}(\ell,\mathbb{C})$ that contains Γ.

(By "virtually connected," we mean that the Zariski closure, although perhaps not connected, has only finitely many components.) Some examples should help to clarify the idea.

Example 1.5. 1) $\mathbb{R}^d$, as embedded in (1) is its own Zariski closure; we say that $\mathbb{R}^d$ is *Zariski closed*.

2) Let

$$G_1 = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \Gamma_1 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Then G_1 is a perfectly natural, connected subgroup, so G_1 is Zariski closed. Because G_1 is the natural connected subgroup that contains Γ_1, we have $\overline{\Gamma_1} = G_1$. (We may say that Γ_1 is *Zariski dense* in G_1.)

3) Let

$$G_2 = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then $\overline{\Gamma_2} = G_2 = \overline{G_2}$.

4) Let

$$G_3 = \left\{ \begin{pmatrix} 1 & t & z \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix} \middle| \begin{array}{c} t \in \mathbb{R}, \\ z \in \mathbb{C} \end{array} \right\} . \tag{2}$$

Then, although G_3 is connected, it is not Zariski closed. The notion of Zariski closure comes from Algebraic Geometry, where only polynomial functions are considered. Thus, because the exponential function is transcendental, not polynomial, an Algebraic Geometer does not see the coupling between the $(1,2)$ entry and the $(3,3)$ entry of the matrix; so, from an Algebraic Geometer's point of view, there is no constraint linking these two matrix entries. The $(1,2)$ entry takes any real value, the $(3,3)$ entry takes any value on the unit circle, and the Zariski closure allows these values entirely independently:

$$\overline{G_3} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{T} \end{pmatrix} .$$

(The analogous example for a topologist would be a discontinuous function $f \colon \mathbb{R} \to \mathbb{T}$, such that the graph of f is dense in $\mathbb{R} \times \mathbb{T}$.) As another important observation, note that $\Gamma_2 \subset G_3$. However, we know that $\overline{\Gamma_2} = G_2 \neq \overline{G_3}$, so Γ_2 is *not* Zariski dense in G_3.

We can now define a version of superrigidity:

Definition 1.6. A discrete subgroup Γ of a Lie group G is *superrigid* in G if every homomorphism $\varphi \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ virtually extends to a continuous homomorphism $\hat{\varphi} \colon G \to \overline{\varphi(\Gamma)}$.

The following superrigidity result (a special case of the main theorem stated later in this section) strengthens Proposition 1.4. Except for the minor discrepancy between extensions and virtual extensions, it also generalizes Proposition 1.1.

Proposition 1.7. *$\mathbb{Z}^k$ is superrigid in $\mathbb{R}^k$.*

A more interesting result (one that deserves to be called a theorem) applies to nonabelian groups. In this section, we consider only solvable groups:

Definition 1.8. A connected Lie subgroup G of $\mathrm{GL}(\ell, \mathbb{C})$ is *solvable* if (perhaps after a suitable change of basis) it is upper triangular.

(This is not the usual definition, but it is more concrete, and it is equivalent to the usual one in our setting (see 4.1).)

For example, the groups G_1, G_2, and G_3 defined in Example 1.5 are obviously solvable. Also, note that any set of pairwise commuting matrices can be simultaneously triangularized, so abelian groups are solvable.

To avoid technical problems that could force us to pass to a finite cover, we usually assume that the fundamental group of G is trivial:

Definition 1.9. A Lie group G is 1-*connected* if it is connected and simply connected.

The following example shows that, for some solvable groups, not all subgroups are superrigid.

Example 1.10. Let

$$G = \begin{pmatrix} \mathbb{R}^+ & \mathbb{R} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} .$$

Then G is obviously solvable, and Γ is a discrete subgroup.

Any homomorphism $G \to \mathbb{R}$ must vanish on the commutator subgroup

$$[G, G] = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \supset \Gamma ,$$

so the only homomorphism $\varphi\colon \Gamma \to \mathbb{R}$ that virtually extends to G is the trivial homomorphism. (Because $\mathbb{R}$ has no nontrivial finite subgroups, any virtually trivial homomorphism into $\mathbb{R}$ must actually be trivial.) Therefore, not all homomorphisms virtually extend, so Γ is not superrigid in G.

The moral of this example is that small subgroups cannot be expected to be superrigid: if Γ is only a small part of G, then a homomorphism defined on Γ knows nothing about most of G, so it cannot be expected to be compatible with the structure of all of G. This suggests that, to obtain a superrigidity theorem, we should assume that Γ is large, in some sense. The correct sense is Zariski density.

Theorem 1.11 (Witte 1997). *If Γ is a discrete subgroup of a* 1*-connected, solvable Lie group $G \subset \mathrm{GL}(\ell, \mathbb{C})$, such that $\overline{\Gamma} = \overline{G}$, then Γ is superrigid in G.*

Example 1.12. Because $\overline{\Gamma_1} = \overline{G_1}$ and $\overline{\Gamma_2} = \overline{G_2}$, the theorem implies that Γ_1 is superrigid in G_1, and Γ_2 is superrigid in G_2. (Actually, the case of G_2 follows already from Proposition 1.7, because it is easy to see that $G_2 \cong \mathbb{R}^3$.) Because $\mathbb{Z}^k$ is Zariski dense in $\mathbb{R}^k$ (for $\mathbb{R}^k$ as in (1)), Proposition 1.7 is a special case of this theorem.

On the other hand, we have $\overline{\Gamma_2} \neq \overline{G_3}$, and, although the theorem does not tell us this, it is easy to see that Γ_2 is not superrigid in G_3. (The subgroup Γ_2 is abelian, and the intersection $\Gamma_2 \cap [G_3, G_3]$ is infinite, so this is much the same as Example 1.10.)

Unfortunately, Proposition 1.1 is not quite a corollary of this theorem, because of the discrepancy between a virtual extension and an actual extension. Section 2 states a version of Theorem 1.11 that, under additional technical hypotheses, provides an actual extension, thus generalizing Proposition 1.1. The section also states a more precise version of Theorem 1.11 that determines exactly which subgroups of a solvable Lie group are superrigid, and briefly discusses superrigidity theorems for Lie groups that are not solvable.

A simple proof of Theorem 1.11 will be given in Sect. 3, modulo an assumption about the existence of syndetic hulls. This gap will be filled in Sect. 5, after some definitions and basic results are recalled from the literature in Sect. 4.

2 Other Superrigidity Theorems

Theorem 1.11 can be extended to stronger results that provide more detailed information about solvable groups, and to broader results that apply to more general groups.

2.1 More on Superrigid Subgroups of Solvable Groups

There are two obvious reasons that the converse of Theorem 1.11 does not hold.

- If Γ is superrigid in B, then $e \times \Gamma$ is superrigid in $A \times B$. So, to be superrigid in a direct product, it suffices to be Zariski dense in one of the factors. This generalizes to semidirect products, as well.
- The group G has many different representations in $\mathrm{GL}(\ell, \mathbb{C})$; it may happen that Γ is Zariski dense in some of these embeddings, but not in others. (For example, if we realize $\mathbb{R}$ as the subgroup of G_3 with $z = 0$, then $\mathbb{Z}$ is not Zariski dense in $\mathbb{R}$.) This ambiguity is eliminated by using only the adjoint representation (even though this is not an embedding if G has a center).

The following corollary shows that these two obvious reasons are the only ones.

Corollary 2.1. *A discrete subgroup Γ of a 1-connected, solvable Lie group G is superrigid if and only if*

1. *$G = A \rtimes B$, for some closed subgroups A and B of G; such that*
2. *B contains a finite-index subgroup Γ' of Γ; and*
3. *$\overline{\mathrm{Ad}_B \Gamma'} = \overline{\mathrm{Ad}_B B}$.*

Proof ($\Rightarrow$). Let $B = \overline{\Gamma}^{\circ}$. The inclusion $\varphi\colon \Gamma \hookrightarrow B$ (virtually) extends to a continuous homomorphism $\hat{\varphi}\colon G \to B$. Since $\hat{\varphi}|_{\Gamma} = \mathrm{Id}_{\Gamma}$, and Γ is Zariski dense in B, it is reasonable to expect that $\hat{\varphi}|_B = \mathrm{Id}_B$. (Actually, this need not quite be true, but it is close to correct.) Then $G = (\ker \hat{\varphi}) \rtimes B$. $\square$

The most important special case is when Γ is a lattice in G:

Definition 2.2. A discrete subgroup Γ of a solvable Lie group G is a *lattice* if G/Γ is compact.

For example, $\mathbb{Z}^k$ is a lattice in $\mathbb{R}^k$, and, in Example 1.5, Γ_1 is a lattice in G_1, and Γ_2 is a lattice in both G_2 and G_3. The superrigidity criterion for lattices is very simple:

Corollary 2.3. *A lattice Γ in a 1-connected, solvable Lie group G is superrigid if and only if $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}_G G}$.*

The following result provides an extension, not just a virtual extension, under mild hypotheses on φ.

Corollary 2.4. *Let Γ be a lattice in a connected, solvable Lie group G, such that $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}_G G}$. If $\varphi\colon \Gamma \to \mathrm{GL}(d, \mathbb{C})$ is a homomorphism, such that*

- *$\varphi(\Gamma) \subset \overline{\varphi(\Gamma)}^{\circ}$,*
- *the center of $\overline{\varphi(\Gamma)}^{\circ}$ is connected, and*
- *$\varphi\big(\Gamma \cap [G, G]\big)$ is unipotent,*

then φ extends to a continuous homomorphism $\hat{\varphi}\colon G \to \overline{\varphi(\Gamma)}$.

The groups G_2 and G_3 of Example 1.5 are non-isomorphic solvable groups that have isomorphic lattices (namely, Γ_2). The following consequence of superrigidity implies that solvable groups with isomorphic lattices differ only by rotations being added to and/or removed from their Zariski closures.

Corollary 2.5. *Suppose*

- *Γ_1 and Γ_2 are lattices in 1-connected, solvable Lie groups G_1 and G_2,*
- *$\overline{\mathrm{Ad}_{G_1} \Gamma_1} = \overline{\mathrm{Ad}_{G_1} G_1}$, and*
- *$\pi\colon \Gamma_1 \to \Gamma_2$ is an isomorphism.*

Then π extends to an embedding $\sigma\colon G_1 \to T \ltimes G_2$, for any maximal compact subgroup T of $\overline{\operatorname{Ad}_{G_2} G_2}^{\circ}$.

Corollary 2.6 (Mostow 1954). *Suppose*

- *Γ_1 and Γ_2 are lattices in 1-connected, solvable Lie groups G_1 and G_2, and*
- *Γ_1 is isomorphic to Γ_2.*

Then G_1/Γ_1 is diffeomorphic to G_2/Γ_2.

2.2 Superrigid Subgroups of Semisimple Groups

For groups that are not solvable, both "lattice" and "superrigid" need to be generalized from the definitions above.

Definition 2.7.
- A discrete subgroup Γ of a Lie group G is a *lattice* if there is G-invariant Borel probability measure on G/Γ.
- A lattice Γ in a Lie group G is *superrigid* if, for every homomorphism $\varphi\colon \Gamma \to \mathrm{GL}(d, \mathbb{C})$, there is a compact, normal subgroup K of $\overline{\varphi(\Gamma)}^{\circ}$, a continuous homomorphism $\hat{\varphi}\colon G \to \overline{\varphi(\Gamma)}^{\circ}/K$, and a finite-index subgroup Γ' of Γ, such that $\hat{\varphi}(\gamma) = \varphi(\gamma)K$, for all $\gamma \in \Gamma'$.

The semisimple case is orders of magnitude more difficult than the solvable case. We still do not have a complete answer, but the following amazing theorem of G. A. Margulis settles most cases.

Theorem 2.8 (Margulis Superrigidity Theorem). *If $n \geq 3$, then every lattice in $\mathrm{SL}(n, \mathbb{R})$ is superrigid.*

The same is true for irreducible lattices in any other connected, semisimple, linear Lie group G with $\mathbb{R}\text{-rank}\, G \geq 2$.

K. Corlette [Cor] proved that lattices in $\mathrm{Sp}(1, n)$ are superrigid, and also lattices in the exceptional group of real rank one. Thus, to complete the study of lattices in semisimple groups, all that remains is to determine which lattices in $\mathrm{SO}(1, n)$ and $\mathrm{SU}(1, n)$ are superrigid. (Many lattices in $\mathrm{SO}(1, n)$ are *not* superrigid.) Lattices are not the whole story, however: H. Bass and A. Lubotzky [B–L] recently constructed an example of a Zariski dense superrigid discrete subgroup Γ of a semisimple group, such that Γ is *not* a lattice.

A superrigidity theorem describes a very close connection between a lattice Γ and the ambient Lie group G. In fact, for semisimple groups, the connection is so close that superrigidity tells us almost exactly what the lattice must be. In all of our examples above, the lattice Γ consists of the integer points of G. The following major consequence of the Margulis Superrigidity Theorem implies that this is essentially the only way to make a lattice in a simple group of higher real rank. (However, one needs to allow certain algebraic integers in place of ordinary integers.)

Corollary 2.9 (Margulis Arithmeticity Theorem). *If $n \geq 3$, then every lattice in $\mathrm{SL}_n(\mathbb{R})$ is arithmetic.*

The same is true for irreducible lattices in any connected, semisimple, linear Lie group G with $\mathbb{R}\text{-rank}\, G \geq 2$.

2.3 Superrigid Subgroups of Other Lie Groups

The following proposition makes it easy to combine the semisimple case with the solvable case.

Proposition 2.10 (L. Auslander). *Let $G = R \rtimes L$ be a Levi decomposition of a connected Lie group G, and let $\sigma\colon G \to L$ be the corresponding quotient map. If Γ is a lattice in G, such that $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}_G G}$, then $\Gamma \cap R$ is a lattice in R, and $\sigma(\Gamma)$ is a lattice in L.*

Corollary 2.11. *Let $G = R \rtimes L$ be a Levi decomposition of a connected, linear Lie group G, and let $\sigma\colon G \to L$ be the corresponding quotient map. A lattice Γ in G is superrigid if and only if*

- *there is a compact, normal subgroup C of $\overline{\mathrm{Ad}_G G}$, such that $\big(\overline{\mathrm{Ad}_G \Gamma}\big)C = \overline{\mathrm{Ad}_G G}$, and*
- *the lattice $\sigma(\Gamma)$ is superrigid in L.*

Pointers to the literature. Corollary 2.1 is from [W2]. Corollaries 2.3, 2.4, 2.5, and 2.11 are from [W1]. (See [Sta] for results related to 2.5, without the compact subgroup T.) Corollary 2.6 and Proposition 2.10 appear in [Rag, Theorems 3.6 and 8.24]. Theorems 2.8 and Corollary 2.9 are discussed in [Mar] and [Zim].

3 Our Prototypical Proof of Superrigidity

We now give a proof of Proposition 1.1 that is somewhat more difficult than necessary, because this argument can be generalized to other groups.

Proof (of Proposition 1.1). Let

- $\hat{\Gamma} = \mathrm{graph}(\varphi) = \big\{ \big(\gamma, \varphi(\gamma)\big) \mid \gamma \in \Gamma \big\} \subset \mathbb{R}^k \times \mathbb{R}^d$,
- $X = \mathrm{span}\, \hat{\Gamma}$ be the subspace of the vector space $\mathbb{R}^k \times \mathbb{R}^d$ spanned by $\hat{\Gamma}$, and
- $p\colon \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^k$ be the natural projection onto the first factor.

Step 1. We have $p(X) = \mathbb{R}^k$. Note that:

- $p(X)$ is connected (because X is connected and p is continuous);
- $p(X)$ is an additive subgroup of $\mathbb{R}^k$ (because X is an additive subgroup, and p is an additive homomorphism); and

- $p(X)$ contains $\mathbb{Z}^k$ (because $p(X)$ contains $p(\hat{\Gamma}) = \operatorname{dom}(\varphi) = \mathbb{Z}^k$).

Since

$$\text{no connected, proper subgroup of } \mathbb{R}^k \text{ contains } \mathbb{Z}^k, \tag{3}$$

the desired conclusion follows.

Step 2. We have $X \cap (0 \times \mathbb{R}^d) = 0$. Because Γ is discrete, we know that φ is continuous, so the φ-image of any compact subset of Γ is compact. This implies that $p|_{\hat{\Gamma}}$, the restriction of p to $\hat{\Gamma}$, is a proper map. (That is, the inverse image of every compact set is compact.) It is a fact that

$$(\operatorname{span} \Lambda)/\Lambda \ \text{ is compact, for every closed subgroup } \Lambda \text{ of } \mathbb{R}^k \times \mathbb{R}^d; \tag{4}$$

therefore, $X = \operatorname{span} \hat{\Gamma}$ differs from $\hat{\Gamma}$ by only a compact amount. Since $p|_{\hat{\Gamma}}$ is proper (and p is a homomorphism), this implies that $p|_X$ is proper. Therefore $X \cap p^{-1}(0)$ is compact. Since p is a homomorphism, we conclude that $X \cap p^{-1}(0)$ is a compact subgroup of $X \cap \mathbb{R}^d$. However,

$$\mathbb{R}^d \text{ has no nontrivial compact subgroups,} \tag{5}$$

so we conclude that $X \cap p^{-1}(0)$ is trivial, as desired.

Step 3. Completion of the proof. From Steps 1 and 2, and the fact that X is a closed subgroup of $\mathbb{R}^k \times \mathbb{R}^d$, we see that X is the graph of a well-defined continuous homomorphism $\hat{\varphi}\colon \mathbb{R}^k \to \mathbb{R}^d$. Also, because $\operatorname{graph}(\varphi) \subset \operatorname{graph}(\hat{\varphi})$, we know that $\hat{\varphi}$ extends φ. □

To generalize this proof to the situation where $\mathbb{Z}^k$, $\mathbb{R}^k$, and $\mathbb{R}^d$ are replaced by more interesting solvable groups Γ, G, and H, we need a closed subgroup X to substitute for the span of $\hat{\Gamma}$. Looking at the proof, we see that the crucial properties of X are that it is a connected subgroup that contains $\hat{\Gamma}$ (so $p(X)$ is a connected subgroup of $\mathbb{R}^k$ that contains $\operatorname{dom}\varphi$ (see Step 1)), and that $X/\hat{\Gamma}$ is compact (see (4)). These properties are captured in the following definition.

Definition 3.1. A *syndetic hull* of a subgroup Γ of a Lie group G is a subgroup X of G, such that X is connected, X contains Γ, and X/Γ is compact.

Thus, the same proof applies in any situation where the following three properties hold:

a. no connected, proper subgroup of G contains Γ (see (3));
b. every closed subgroup $\hat{\Gamma}$ of $G \times H$ has a syndetic hull (see (4)); and
c. H has no nontrivial compact subgroups (see (5)).

Two of these properties pose little difficulty:

(a) If Γ is a lattice in a 1-connected, solvable Lie group G, then no connected, proper subgroup of G contains Γ (see 4.3(2)).
(c) If H is a 1-connected, solvable Lie group, then H has no nontrivial compact subgroups (see 4.3(3)).

However, Property (b) may fail, as is illustrated by the following example.

Example 3.2. Let

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset G_3 \qquad \text{and} \qquad S = \left\{ \begin{pmatrix} 1 & t & \mathbb{R} \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

(see (2)). Then S is the only reasonable candidate to be a syndetic hull of Γ in G_3. However, S is not closed under multiplication, so it is not a subgroup of G_3. Thus, one sees that Γ does not have a syndetic hull in G_3.

The upshot is that proving superrigidity (in the setting of solvable groups) reduces to the problem of showing that syndetic hulls exist. It turns out that Zariski dense subgroups always have a syndetic hull. (The reader can easily verify that the subgroup Γ of Example 3.2 is not Zariski dense in G_3.) However, the following key result (which will be proved in Sect. 5) shows that a much weaker hypothesis suffices: $\overline{\mathrm{Ad}_G \Gamma}$ need only contain a maximal compact subgroup, not all of $\overline{\mathrm{Ad}_G G}$.

Theorem 3.3. *If Γ is a closed subgroup of a connected, solvable Lie group G, such that*

$$\overline{\mathrm{Ad}_G \Gamma} \text{ contains a maximal compact subgroup of } \overline{\mathrm{Ad}_G G}^{\circ}, \tag{6}$$

then Γ has a syndetic hull in G. Furthermore, if G is 1-connected, then the syndetic hull is unique.

For example, if G is a 1-connected, $\mathbb{R}$-split solvable group (that is, if G is an upper triangular subgroup of $\mathrm{GL}(n,\mathbb{R})$), then $\overline{\mathrm{Ad}_G G}$ has no compact subgroups, so the hypothesis of the theorem is trivially satisfied.

Corollary 3.4. *In a 1-connected, $\mathbb{R}$-split solvable group, syndetic hulls exist and are unique.*

So the proof applies:

Proposition 3.5. *Suppose*

- *G_1 and G_2 are 1-connected, $\mathbb{R}$-split solvable groups,*
- *Γ_1 is a lattice in G_1, and*
- *$\varphi\colon \Gamma_1 \to G_2$ is a homomorphism.*

Then φ extends uniquely to a continuous homomorphism $\hat{\varphi}\colon G_1 \to G_2$.

Corollary 3.6 (Saito 1957). *Suppose Γ_i is a lattice in a 1-connected, $\mathbb{R}$-split solvable group G_i, for $i = 1, 2$. If $\Gamma_1 \cong \Gamma_2$, then $G_1 \cong G_2$.*

Let us now use Theorem 3.3 to prove a superrigidity theorem.

Proof (of Theorem 1.11). We are given a homomorphism $\varphi\colon \Gamma \to \mathrm{GL}(d, \mathbb{C})$.

Case 1. Assume Γ is a lattice in G. Let

- $H = \overline{\varphi(\Gamma)}$,
- $\hat{G} = G \times H$,
- $\hat{\Gamma} = \mathrm{graph}(\varphi) \subset G \times H$, and
- $p\colon G \times H \to G$ be the natural projection onto the first factor.

We use the proof of Proposition 1.1, so there are only two issues to address. First, we need to show that $\hat{\Gamma}$ has a syndetic hull X in $\hat{G}$. Second, because H may not be 1-connected, we do not have property (c), the analogue of (5).

Recall that Zariski closures are virtually connected. (This is stated formally in Lemma 4.7 below.) Hence, H has only finitely many components, so, by passing to a finite-index subgroup of Γ, we may assume that $\varphi(\Gamma) \subset H^\circ$, so $\hat{\Gamma} \subset G \times H^\circ = \hat{G}^\circ$.

By assumption, $\overline{\Gamma}^\circ$ contains a maximal compact subgroup S of $\overline{G}^\circ$, and, by definition, $\overline{\varphi(\Gamma)}^\circ$ contains a maximal compact subgroup T of H°. Therefore, the projection of $\overline{\hat{\Gamma}}^\circ$ to each factor of $\overline{G}^\circ \times H^\circ$ contains a maximal compact subgroup of that factor. However, $\overline{\hat{\Gamma}}$ is diagonally embedded in $\overline{G} \times H$, so it probably does not contain the product $S \times T$, which is a maximal compact subgroup of $\overline{G}^\circ \times H^\circ$. Thus, Theorem 3.3 probably does not apply directly. However, $S \times T$ is contained in $\overline{\hat{\Gamma}} T$, so the rather technical Theorem 3.7 below, which can be proved in almost exactly the same way as Theorem 3.3, does apply. So we conclude that some finite-index subgroup of $\hat{\Gamma}$ has a syndetic hull X in $\hat{G}^\circ$, as desired. (Note that, because $\mathrm{graph}(\hat{\varphi}) = X$ contains a finite-index subgroup of $\hat{\Gamma}$, the homomorphism $\hat{\varphi}$ virtually extends φ.)

Theorem 3.7 asserts that we may take the syndetic hull X to be simply connected; thus, X has no nontrivial compact subgroups. Hence, the subgroup $X \cap p^{-1}(e)$ also has no compact subgroups. Assumption (5) was used only to obtain this conclusion, so we have no need for (c).

Case 2. The general case. From Theorem 3.3, we know that Γ has a syndetic hull B. So Γ is a lattice in B, and, by assumption, $\overline{\Gamma} = \overline{G} \supset \overline{B}$. Therefore, Case 1 implies that φ virtually extends to a continuous homomorphism $\varphi^*\colon B \to \mathrm{GL}(d, \mathbb{C})$.

Now, because B is connected, and $\overline{B} = \overline{G}$, one can show that $[G, G] \subset B$. So it is not hard to extend φ^* to a continuous homomorphism $\hat{\varphi}\colon G \to \mathrm{GL}(d, \mathbb{C})$. □

Theorem 3.7. *Let Γ be a discrete subgroup of a connected, solvable, linear Lie group G.*

If there is a compact subgroup S of $\overline{\Gamma}$ and a compact subgroup T of G, such that ST is a maximal compact subgroup of $\overline{G}^{\circ}$, then some finite-index subgroup Γ' of Γ has a simply connected syndetic hull in G.

Pointers to the literature. Definition 3.1 is slightly modified from [F–G]. (In our terminology, they proved that every solvable subgroup Γ of $\mathrm{GL}(\ell, \mathbb{C})$ virtually has a syndetic hull in $\overline{\Gamma}$.) Theorem 3.3 appears in [W1]. For the special case where G_1 and G_2 are nilpotent, Corollary 3.6 was proved by Malcev, and this special case appears in [Rag, Theorem 2.11, p. 33]. Theorem 3.7 is from [W3].

4 Solvable Lie Groups and Zariski Closed Subgroups

We now recall (without proof) some rather standard results on solvable Lie groups and Zariski closures.

4.1 Solvable Lie Groups and Their Subgroups

Remark 4.1. Although the definition of "solvable" given in Defn. 1.8 is not the usual one, the Lie–Kolchin Theorem [Hum, Theorem 17.6, pp. 113–114] implies that a connected subgroup G of $\mathrm{GL}(\ell, \mathbb{C})$ satisfies (1.8) if and only if it is solvable in the usual sense. Thus, this naive description is adequate for our purposes.

Also, Ado's Theorem [Var, Theorem 3.18.16, pp. 246–247] implies that every 1-connected, solvable Lie group is isomorphic to a closed subgroup of some $\mathrm{GL}(\ell, \mathbb{C})$, so there is no loss of generality in restricting our attention to linear groups.

The following observation is immediate from the usual definition of solvability:

Lemma 4.2. *If G is a nontrivial, connected, solvable Lie group, then*

$$\dim[G, G] < \dim G .$$

Proposition 4.3. *Let H be a connected subgroup of a 1-connected, solvable Lie group G.*

1. *H is closed, simply connected, and diffeomorphic to some $\mathbb{R}^d$;*
2. *If G/H is compact, then $H = G$;*
3. *If C is a compact subgroup of G, then C is trivial.*

Lemma 4.4. *Let Q be a closed subgroup of a connected, solvable group G.*

1. *If G/Q is simply connected, then Q is connected, and Q contains a maximal compact subgroup of G;*
2. *If Q has only finitely many components, and Q contains a maximal compact subgroup of G, then Q is connected, and G/Q is simply connected.*

Lemma 4.5. *If G is any Lie group with only finitely many connected components, then*

1. *G has a maximal compact subgroup, and*
2. *all maximal compact subgroups of G are conjugate to each other.*

4.2 Zariski Closed Subgroups of $\mathbf{GL}(\boldsymbol{\ell}, \mathbb{C})$

The following definition formalizes the idea that a subgroup is Zariski closed if it is defined by polynomial functions. Also, we are thinking of $\mathrm{GL}(\ell, \mathbb{C})$ as being a real variety of dimension $2\ell^2$, rather than a complex variety of dimension ℓ^2.

Definition 4.6. A subset X of $\mathbb{R}^N$ is *Zariski closed* if there is a (finite or infinite) collection $\{P_k\}$ of real polynomials in N variables, such that

$$X = \{\, x \in \mathbb{R}^N \mid P_k(x) = 0, \text{ for all } k \,\} .$$

Let $\varphi\colon \mathrm{GL}(\ell, \mathbb{C}) \to \mathbb{R}^{\ell^2+2}$ be the identification of $\mathrm{GL}(\ell, \mathbb{C})$ with a (Zariski closed) subset of $\mathbb{R}^{\ell^2+2}$ given by listing the real and imaginary parts of the determinant and of each matrix entry:

$$\varphi(g) = \big(\Re(\det g), \Im(\det g), \Re g_{1,1}, \Im g_{1,1}, \Re g_{1,2}, \Im g_{1,2}, \dots , \Re g_{\ell,\ell}, \Im g_{\ell,\ell}\big) .$$

A subgroup Q of $\mathrm{GL}(\ell, \mathbb{C})$ is *Zariski closed* if $\varphi(Q)$ is a Zariski closed subset of $\mathbb{R}^{\ell^2+2}$.

The *Zariski closure* $\overline{\Gamma}$ of a subgroup Γ of $\mathrm{GL}(\ell, \mathbb{C})$ is the (unique) smallest Zariski closed subgroup that contains Γ.

Lemma 4.7. *Any Zariski closed subgroup has only finitely many connected components.*

Lemma 4.8. *Let H be a Zariski-closed subgroup of* $\mathrm{GL}(\ell, \mathbb{C})$.

1. *For any subgroup Γ of H, the centralizer $C_H(\Gamma)$ is Zariski closed;*
2. *For any connected subgroup U of H, the normalizer $N_H(U)$ is Zariski closed.*

Corollary 4.9. *If G is a connected subgroup of* $\mathrm{GL}(\ell, \mathbb{C})$*, then $\overline{G}$ normalizes G.*

Theorem 4.10 (Borel Density Theorem). *If Γ is a closed subgroup of a connected, solvable Lie group G, such that G/Γ is compact, then $\overline{\mathrm{Ad}_G \Gamma T} = \overline{\mathrm{Ad}_G G}$, for every maximal compact subgroup T of $\overline{\mathrm{Ad}_G G}^\circ$.*

Corollary 4.11. *Suppose Γ is a closed subgroup of a connected, solvable Lie group G, such that G/Γ is compact. If (6) holds, then $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}_G G}$.*

Pointers to the literature. Propositions 4.3(1) and 4.3(3) and Lemma 4.5 appear in [Hoc, Theorems 12.2.2, 12.2.3, and 15.3.1]. Proposition 4.3(2) is due to G. D. Mostow [Mos, Proposition 11.2]. Lemma 4.4 follows from the homotopy long exact sequence of the fibration $Q \to G \to G/Q$; cf. [W1, Lemma 2.17]. Lemma 4.7 appears in [P–R, Theorem 3.6]. Lemma 4.8(1) follows from [Hum, Prop 8.2b]. Lemma 4.8(2) follows from the proof of [Zim, Theorem 3.2.5]. A generalization of Theorem 4.10 appears in [Dan, Corollary 4.2].

5 Existence of Syndetic Hulls

Constructing a syndetic hull requires some way to show that a subgroup is connected. The following result on intersections of connected subgroups is our main tool in this regard.

Proposition 5.1. *Let G and Q be solvable Lie subgroups of $\mathrm{GL}(\ell, \mathbb{C})$. If*

- *G is connected,*
- *Q is Zariski closed (or, more generally, Q has finite index in $\overline{Q}$), and*
- *Q contains a maximal compact subgroup of $\overline{G}^\circ$,*

then $G \cap Q$ is connected.

Proof. Let T be a maximal compact subgroup of $\overline{G}^\circ$ that is contained in Q.

Case 1. Assume $Q \subset \overline{G}^\circ$. Because $\overline{G}$ normalizes G (see 4.9), we know that Q normalizes G, so GQ is a subgroup of $\mathrm{GL}(\ell, \mathbb{C})$. Since Q contains the maximal compact subgroup T of GQ, we see that GQ/Q is simply connected (see 4.4(2)). Hence $G/(G \cap Q) \simeq GQ/Q$ is simply connected, so $G \cap Q$ is connected (see 4.4(1)).

Case 2. The general case. Because $\overline{G}^\circ \cap Q$ contains the maximal compact subgroup T of $\overline{G}^\circ$, Case 1 implies that

$$G \cap Q = (G \cap \overline{G}^\circ) \cap Q = G \cap (\overline{G}^\circ \cap Q)$$

is connected, as desired. □

The following corollary is obtained by using the proposition to show that $\hat{G} \cap \hat{Q}$ is connected, where $\hat{G} = \mathrm{graph}(\rho)$ and $\hat{Q} = \overline{G} \times Q$.

Corollary 5.2. *Let*

- *G be a connected, solvable Lie group,*
- *$\rho\colon G \to \mathrm{GL}(d,\mathbb{C})$ be a finite-dimensional, continuous representation, and*
- *Q be a Zariski closed subgroup of $\mathrm{GL}(d,\mathbb{C})$, such that Q contains a maximal compact subgroup of $\overline{\rho(G)}^{\circ}$.*

Then $\rho^{-1}(Q)$ is connected.

Corollary 5.3. *Let Γ be a closed subgroup of a 1-connected, solvable Lie group G, such that (6) holds.*

1. *If $\Gamma \subset Z(G)$, then $Z(G)$ is connected;*
2. *If Γ is abelian, then $C_G(\Gamma)$ is connected;*
3. *If U is any connected subgroup of G that is normalized by Γ, then $N_G(U)$ is connected.*

Proof. To simplify the notation, let us assume $\overline{\Gamma}$ contains a maximal compact subgroup of $\overline{G}^{\circ}$, ignoring the adjoint representation. (Without this simplification, the proof would use Corollary 5.2, with $\rho = \mathrm{Ad}_G$, instead of using Proposition 5.1, as we do here.) Under this assumption, Proposition 5.1 implies that if Q is any Zariski closed subgroup of $\mathrm{GL}(\ell,\mathbb{C})$ that contains Γ, then $G \cap Q$ is connected. We apply this fact with:
(1) $Q = C_{\overline{G}}(G)$ (see 4.8(1)); so $Z(G) = C_G(G) = G \cap Q$ is connected.
(2) $Q = C_{\overline{G}}(\Gamma)$ (see 4.8(1)); so $C_G(\Gamma) = G \cap Q$ is connected.
(3) $Q = N_{\overline{G}}(U)$ (see 4.8(2)); so $N_G(U) = G \cap Q$ is connected. □

Theorem 5.4. *Suppose Γ is a closed subgroup of a 1-connected, solvable Lie group G. If (6) holds, then Γ has a unique syndetic hull in G.*

Proof. Let us first prove uniqueness: suppose S_1 and S_2 are syndetic hulls of Γ. We have $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}_G S_i}$ for $i = 1,2$ (see 4.11); so $\overline{\mathrm{Ad}_G S_1} = \overline{\mathrm{Ad}_G S_2}$. Therefore S_1 and S_2 normalize each other (see 4.9), so S_1S_2 is a subgroup of G. It is simply connected (see 4.3(1)), and $S_1S_2/S_2 \simeq S_1/(S_1 \cap S_2)$ is compact (because $\Gamma \subset S_1 \cap S_2$), so Lemma 4.3(2) implies that $S_2 = S_1S_2$; thus $S_1 \subset S_2$. Similarly, $S_2 \subset S_1$. Therefore $S_1 = S_2$, so the syndetic hull, if it exists, is unique.

We now prove existence. We may assume, by induction on $\dim G$ (see 4.2), that $\Gamma \cap [G,G]$ has a unique syndetic hull U in $[G,G]$ (note that $\overline{\mathrm{Ad}_G[G,G]}$ is unipotent, so it has no nontrivial compact subgroups).

Case 1. Assume G is abelian. Because Γ is a normal subgroup of G, we may consider the quotient group G/Γ: let K/Γ be a maximal compact subgroup of G/Γ. By definition, K/Γ is compact. Also, $G/K \simeq (G/\Gamma)/(K/\Gamma)$ is simply connected (see 4.4(2)), so K is connected (see 4.4(1)). Therefore K is a syndetic hull of Γ.

Case 2. Assume $\Gamma \subset Z(G)$. Because $Z(G)$ is connected (see 5.3(1)), we know, from Case 1, that Γ has a syndetic hull S in $Z(G)$. Then S is a syndetic hull of Γ in G.

Case 3. Assume Γ is abelian. We have $\Gamma \subset C_G(\Gamma)$, and $C_G(\Gamma)$ is connected (see 5.3(2)), so, from Case 2, we know that Γ has a syndetic hull S in $C_G(\Gamma)$. Then S is also a syndetic hull of Γ in G.

Case 4. Assume U is a normal subgroup of G. Let $\tau \colon G \to G/U$ be the natural homomorphism. Then, because $U/(\Gamma \cap U)$ is compact, we see that ΓU is closed, so $\tau(\Gamma)$ is a closed subgroup of $\tau(G)$. Also, because $[\Gamma, \Gamma] \subset U$, we have

$$[\tau(\Gamma), \tau(\Gamma)] = \tau\big([\Gamma, \Gamma]\big) \subset \tau(U) = e \ ,$$

so $\tau(\Gamma)$ is abelian. Thus, from Case 3, we know that $\tau(\Gamma)$ has a syndetic hull S in $\tau(G)$. Then $\tau^{-1}(S)$ is a syndetic hull of Γ in G.

Case 5. The general case. The uniqueness of the syndetic hull U implies that Γ normalizes U; that is, $\Gamma \subset N_G(U)$. Now $N_G(U)$ is connected (see 5.3(3)), so, from Case 4, we know that Γ has a syndetic hull S in $N_G(U)$; then S is also a syndetic hull of Γ in G. □

Corollary 5.5. *Suppose Γ is a closed subgroup of a connected, solvable Lie group G. If (6) holds, then Γ has a syndetic hull in G.*

Proof. Write $G = \widetilde{G}/Z$ and $\Gamma = \widetilde{\Gamma}/Z$, where Z is some discrete, normal subgroup of the center of the universal cover $\widetilde{G}$ of G. If S is any syndetic hull of $\widetilde{\Gamma}$, then S/Z is a syndetic hull of Γ. □

Remark 5.6. If G is not simply connected, then syndetic hulls may not be unique. (For example, e and $\mathbb{T}$ are two syndetic hulls of e in $\mathbb{T}$.)

Proposition 5.1 has the following corollary. Theorem 3.7 is proved almost exactly the same way as Theorem 5.4, but using this corollary in place of Proposition 5.1. However, a small additional argument is needed when G is abelian, to show that the syndetic hull can be chosen to be simply connected in this base case.

Corollary 5.7. *Let G and Q be solvable Lie subgroups of $\mathrm{GL}(\ell, \mathbb{C})$. If*

- *G is connected,*
- *Q is Zariski closed, and*
- *there are compact subgroups S of Q and T of $\overline{G}$, such that ST is a maximal compact subgroup of $\overline{G}^\circ$,*

then $G \cap Q$ is virtually connected.

Proof. By replacing Q with $\overline{G} \cap Q$, we may assume $Q \subset \overline{G}$. Also, by replacing T with a subgroup, we may assume $S \cap T$ is finite. From the structure theory of solvable Zariski closed subgroups [Hum, Theorems 19.3 and 34.3b], we have $\overline{G}^\circ = (ST) \ltimes V$ and $Q^\circ = S \ltimes (Q \cap V)$, where V is the subgroup generated by the elements of $\overline{G}$, all of whose eigenvalues are real and positive; then, because Q contains the maximal compact subgroup S of $SV = \left(\overline{G \cap SV}\right)^\circ$, Proposition 5.1 implies $Q \cap \big(G \cap (SV)\big)^\circ$ is connected. This is a finite-index subgroup of $Q \cap G$, because $Q^\circ \subset SV$, and $G \cap (SV)$ is virtually connected. Therefore, $Q \cap G$ is virtually connected. □

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